X.—*The Relations between the Coaxial Minors of a Determinant of the Fourth Order.* By THOMAS MUIR, LL.D.

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1. The existence of relations between the coaxial minors of a determinant was first discovered by MACMAHON in 1893. The whole literature of the subject is comprised in three papers, viz.:—

MACMAHON, *Phil. Trans.*, clxxxv. pp. 111–160.
MUIR, *Phil. Mag.*, 5th series, xli. pp. 537–541.
NANSON, *Phil. Mag.*, 5th series, xliv. pp. 362–367.

My present object is to continue the investigation of the relations in question, and more particularly to draw attention to an *explicit* expression for a determinant of the 4th order in terms of its own coaxial minors. At the outset some fresh considerations regarding determinants in general will be found useful.

2. As is well known, the coaxial minors of a determinant of the nth order are $2^n - 1$ in number, the determinant itself and each of the elements of its primary diagonal being counted. For example, the coaxial minors of $|a_1b_2c_3d_4|$ are

$$\begin{array}{llllll} |a_1b_2c_3d_4|, \\ |a_1b_2c_3|, & |a_1b_2d_4|, & |a_1c_3d_4|, & |b_2c_3d_4|, \\ |a_1b_2|, & |a_1c_3|, & |a_1d_4|, & |b_2c_3|, & |b_2d_4|, & |c_3d_4|, \\ a_1, & b_2, & c_3, & d_4. \end{array}$$

Of these the first $2^n - 1 - n$ may be devertebrated, if we may say so, by substituting zeros for the elements of their primary diagonals; and the determinants thus resulting are found to be of considerable interest. They appear in CAYLEY's well-known expansion-theorem, which for a determinant of the 3rd order is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} + a_1\begin{vmatrix} \cdot & b_3 \\ c_2 & \cdot \end{vmatrix} + b_2\begin{vmatrix} \cdot & a_3 \\ c_1 & \cdot \end{vmatrix} + c_3\begin{vmatrix} \cdot & a_2 \\ b_1 & \cdot \end{vmatrix} + a_1b_2c_3.$$

Indeed this theorem may be described as giving an expression for a determinant in terms of its own devertebrated coaxial minors and its primary diagonal elements.

Now, if we use CAYLEY's expansion in connection with each of the first $2^n - 1 - n$ coaxial minors, we obtain $2^n - 1 - n$ equations, linear in respect to the devertebrated minors. So that, on solving for the latter, there must result an expression for each devertebrated coaxial minor in terms of the vertebrate coaxial minors and the primary diagonal elements. The general theorem thus obtained is

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 & \cdots \\ b_1 & \cdot & b_3 & b_4 & \cdots \\ c_1 & c_2 & \cdot & c_4 & \cdots \\ d_1 & d_2 & d_3 & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = |a_1b_2c_3d_4\ldots| - \Sigma a_1|b_2c_3d_4\ldots| + \Sigma a_1b_2|c_3d_4\ldots| + \ldots \tag{A}$$

It may be viewed as a sort of converse of CAYLEY'S, which in outward form it very closely resembles.

3. The truth of it may be established by proceeding in the manner just indicated; but there is another available process which has the advantage of presenting it merely as the ultimate case of a more general theorem, viz., a theorem for similarly expanding a determinant which is only *partially* devertebrated.

Taking determinants of the 3rd order, we have in succession and without any difficulty of verification,

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3|,$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| + a_1b_2c_3,$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| - c_3|a_1b_2| + 2a_1b_2c_3.$$

Proceeding to the 4th order, we have with equal simplicity in the first case

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = |a_1b_2c_3d_4| - a_1|b_2c_3d_4|. \tag{A$_1$}$$

For the next case we have similarly

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - b_2\begin{vmatrix} \cdot & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix},$$

and as each of the determinants on the right has already been expanded in the new form, there is at once obtained by substitution

$$|a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| + a_1b_2|c_3d_4|. \qquad (\text{A}_2$$

Again, we have

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - c_3 \begin{vmatrix} \cdot & a_2 & a_4 \\ b_1 & \cdot & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix},$$

$$\begin{aligned} &= |a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| + a_1b_2|c_3d_4| \\ &\qquad - c_3\{|a_1b_2d_4| - a_1|b_2d_4| - b_2|a_1d_4| + a_1b_2d_4\}, \\ &= |a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| - c_3|a_1b_2d_4| \\ &\qquad + a_1b_2|c_3d_4| + a_1c_3|b_2d_4| + b_2c_3|a_1d_4| \\ &\qquad - a_1b_2c_3d_4. \end{aligned} \tag{A_3}$$

And lastly, by proceeding in exactly the same way, we have the theorem of the preceding section, viz.:—

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} = |a_1b_2c_3d_4| - \Sigma a_1|b_2c_3d_4| + \Sigma a_1b_2|c_3d_4| - 3a_1b_2c_3d_4,$$

where the Σ refers to combinations of the four elements, a_1, b_2, c_3, d_4.

4. MACMAHON's problem of expressing the determinant of the 4th order in terms of its coaxial minors may thus be transformed into something apparently simpler, viz., expressing the determinant in terms of its *devertebrated* coaxial minors and the primary diagonal elements.

In the case of the determinant $|a_1b_2c_3d_4|$ the eleven (*i.e.*, $2^4 - 1 - 4$) devertebrated coaxial minors are

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_1c_4d_3 + a_3b_4c_1d_2 + a_4b_3c_2d_1 - \begin{Bmatrix} a_2b_4c_1d_3 + a_3b_1c_4d_2 \\ + a_2b_3c_4d_1 + a_4b_1c_2d_3 \\ + a_4b_3c_1d_2 + a_3b_4c_2d_1 \end{Bmatrix} = \text{D say},$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_3c_1 + a_3b_1c_2 = \text{C}_1 \text{ say},$$

$$\begin{vmatrix} \cdot & a_2 & a_4 \\ b_1 & \cdot & b_4 \\ d_1 & d_2 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_4d_1 + a_4b_1d_2 = \text{C}_2 \text{ say},$$

$$\begin{vmatrix} \cdot & a_3 & a_4 \\ c_1 & \cdot & c_4 \\ d_1 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad a_3c_4d_1 + a_4c_1d_3 = \text{C}_3 \text{ say},$$

$$\begin{vmatrix} \cdot & b_3 & b_4 \\ c_2 & \cdot & c_4 \\ d_2 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad b_3c_4d_2 + b_4c_2d_3 = \text{C}_4 \text{ say},$$

$$\begin{vmatrix} \cdot & a_2 \\ b_1 & \cdot \end{vmatrix} \quad i.e., \ -a_2b_1 = \mathrm{B}_1 \text{ say,}$$

$$\begin{vmatrix} \cdot & a_3 \\ c_1 & \cdot \end{vmatrix} \quad i.e., \ -a_3c_1 = \mathrm{B}_2 \text{ say,}$$

$$\begin{vmatrix} \cdot & a_4 \\ d_1 & \cdot \end{vmatrix} \quad i.e., \ -a_4d_1 = \mathrm{B}_3 \text{ say,}$$

$$\begin{vmatrix} \cdot & b_3 \\ c_2 & \cdot \end{vmatrix} \quad i.e., \ -b_3c_2 = \mathrm{B}_4 \text{ say,}$$

$$\begin{vmatrix} \cdot & b_4 \\ d_2 & \cdot \end{vmatrix} \quad i.e., \ -b_4d_2 = \mathrm{B}_5 \text{ say,}$$

$$\begin{vmatrix} \cdot & c_4 \\ d_3 & \cdot \end{vmatrix} \quad i.e., \ -c_4d_3 = \mathrm{B}_6 \text{ say.}$$

Using the last six equations to eliminate b_1, c_1, d_1, c_2, d_2, d_3—these being the elements on one side of the primary diagonal of $|a_1b_2c_3d_4|$—from the preceding five equations, we have

$$\left.\begin{aligned}
\mathrm{B}_1\mathrm{B}_6+\mathrm{B}_2\mathrm{B}_5+\mathrm{B}_3\mathrm{B}_4-\left[\begin{matrix} \mathrm{B}_2\mathrm{B}_6\dfrac{a_2b_4}{a_3c_4} + \mathrm{B}_1\mathrm{B}_5\dfrac{a_3c_4}{a_2b_4} \\ -\ \mathrm{B}_3\dfrac{a_2b_3c_4}{a_4} - \mathrm{B}_1\mathrm{B}_4\mathrm{B}_6\dfrac{a_4}{a_2b_3c_4} \\ +\mathrm{B}_2\mathrm{B}_5\dfrac{a_4b_3}{a_3b_4} + \mathrm{B}_3\mathrm{B}_4\dfrac{a_3b_4}{a_4b_3} \end{matrix}\right] &= \mathrm{D}, \\
-\ \mathrm{B}_2\frac{a_2b_3}{a_3} + \mathrm{B}_1\mathrm{B}_4\frac{a_3}{a_2b_3} &= \mathrm{C}_1, \\
-\ \mathrm{B}_3\frac{a_2b_4}{a_4} + \mathrm{B}_1\mathrm{B}_5\frac{a_4}{a_2b_4} &= \mathrm{C}_2, \\
-\ \mathrm{B}_3\frac{a_3c_4}{a_4} + \mathrm{B}_2\mathrm{B}_6\frac{a_4}{a_3c_4} &= \mathrm{C}_3, \\
-\ \mathrm{B}_5\frac{b_3c_4}{b_4} + \mathrm{B}_4\mathrm{B}_6\frac{b_4}{b_3c_4} &= \mathrm{C}_4.
\end{aligned}\right\}$$

But the four fractional quantities $\dfrac{a_2b_3}{a_3}, \dfrac{a_2b_4}{a_4}, \dfrac{a_3c_4}{a_4}, \dfrac{b_3c_4}{b_4}$ — or say $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ — in the last four equations are connected by the relation

$$\gamma_1\gamma_3 = \gamma_2\gamma_4,$$

and the three similar quantities in the remaining equation of the set are expressible in terms of these four, viz.:—

$$\frac{a_2b_4}{a_3c_4} = \frac{\gamma_2}{\gamma_3} \quad\text{or}\quad \frac{\gamma_1}{\gamma_4},$$

$$\frac{a_2b_3c_4}{a_4} = \gamma_1\gamma_3 \quad\text{or}\quad \gamma_2\gamma_4,$$

$$\frac{a_4b_3}{a_3b_4} = \frac{\gamma_1}{\gamma_2} \quad\text{or}\quad \frac{\gamma_4}{\gamma_3}.$$

It is thus possible by the elimination of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ to deduce five equations, not more than two of which, however, can be independent.

5. Taking the first four equations of the set of five, and using γ_1, γ_2, γ_3 as just indicated, we have

$$\left.\begin{aligned}
B_1B_6+B_2B_5+B_3B_4-D &= \left(B_2B_6\frac{\gamma_2}{\gamma_3}+B_1B_5\frac{\gamma_3}{\gamma_2}\right)-\left(B_3\gamma_1\gamma_3+B_1B_4B_6\frac{1}{\gamma_1\gamma_3}\right)+\left(B_2B_5\frac{\gamma_1}{\gamma_2}+B_3B_4\frac{\gamma_2}{\gamma_1}\right)\\
C_1 &= -B_2\gamma_1+B_1B_4\frac{1}{\gamma_1},\\
C_2 &= -B_3\gamma_2+B_1B_5\frac{1}{\gamma_2},\\
C_3 &= -B_3\gamma_3+B_2B_6\frac{1}{\gamma_3}.
\end{aligned}\right\}$$

Now, by means of each pair of the last three of these equations, the γ's may be eliminated from a corresponding one of the bracketed expressions in the first equation, the results of this action in fact being

$$\begin{aligned}
B_2B_6\frac{\gamma_2}{\gamma_3}+B_1B_5\frac{\gamma_3}{\gamma_2} &= \frac{-C_2C_3+\sqrt{C_2^2+4B_1B_3B_5}\sqrt{C_3^2+4B_2B_3B_6}}{2B_3},\\
B_3\gamma_1\gamma_3+B_1B_4B_6\frac{1}{\gamma_1\gamma_3} &= \frac{C_1C_3+\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_3^2+4B_2B_3B_6}}{2B_2},\\
B_2B_5\frac{\gamma_1}{\gamma_2}+B_3B_4\frac{\gamma_2}{\gamma_1} &= \frac{-C_1C_2+\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_2^2+4B_1B_3B_5}}{2B_1}.
\end{aligned}$$

We thus have

$$\begin{aligned}
D=B_1B_6+B_2B_5+B_3B_4+\frac{C_2C_3}{2B_3}+\frac{C_3C_1}{2B_2}+\frac{C_1C_2}{2B_1}&\\
-\frac{1}{2B_3}\sqrt{C_2^2+4B_1B_3B_5}\sqrt{C_3^2+4B_2B_3B_6}+\frac{1}{2B_2}\sqrt{C_3^2+4B_2B_3B_6}\sqrt{C_1^2+4B_1B_2B}&\\
-\frac{1}{2B_1}\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_2^2+4B_1B_3B_5},&
\end{aligned}$$

—a relation among ten of the eleven devertebrated coaxial minors of $|a_1b_2c_3d_4|$. Then as for each of the ten there is an expression in terms of the vertebrate coaxial minors, and, in the case of one of them, viz., D, this expression involves the original determinant $|a_1b_2c_3d_4|$, it is clear that we may deduce from this the result foreshadowed by MACMAHON, viz., an expression for $|a_1b_2c_3d_4|$ in terms of its coaxial minors.

Making the actual substitutions in places where subsequent simplification is readily possible,* we find

$$\begin{aligned}
|a_1b_2c_3d_4| = \Sigma a_1|b_2c_3d_4| + \Sigma|a_1b_2|\,|c_3d_4| - 2\Sigma a_1b_2|c_3d_4| + 6a_1b_2c_3d_4 + \frac{C_1C_2}{2B_1} + \frac{C_1C_3}{2B_2} + \frac{C_2C_3}{2B_3}&\\
-\frac{1}{2B_1}\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_2^2+4B_1B_3B_5}+\frac{1}{2B_2}\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_3^2+4B_2B_3B_6}&\\
-\frac{1}{2B_3}\sqrt{C_2^2+4B_1B_3B_5}\sqrt{C_3^2+4B_2B_3B_6},&
\end{aligned}$$

* In the case of each expression under a root-sign a certain amount of simplification is also possible, *e.g.*, we find

$$\begin{aligned}
C_2^2+4B_1B_3B_5 = |a_1b_2d_4|^2 - \Sigma 2|a_1b_2d_4|\,|b_2d_4|a_1 + 4|a_1b_2d_4|a_1b_2d_4 + 4|a_1b_2|\,|a_1d_4|\,|b_2d_4|&\\
+\Sigma a_1^2|b_2d_4|^2 - \Sigma 2a_1b_2|a_1d_4|\,|b_2d_4|.&
\end{aligned}$$

where $B_1, B_2, \ldots B_6, C_1, C_2, C_3$ have the significations given to them in section 4, but are to be replaced by using the theorem of sections 2, 3.

6. Again, taking the last four of the set of five equations in section 4, and bearing in mind that $\gamma_1\gamma_3 = \gamma_2\gamma_4$, all that is necessary for elimination is to put

$$\begin{aligned}
\gamma_1 &= \frac{-C_1 + \sqrt{C_1^2 + 4B_1B_2B_4}}{2B_2} \quad \text{or} \quad \frac{2B_1B_4}{C_1 + \sqrt{C_1^2+4B_1B_2B_4}}, \\
\gamma_2 &= \frac{-C_2 + \sqrt{C_2^2 + 4B_1B_3B_5}}{2B_3} \quad \text{or} \quad \frac{2B_1B_5}{C_2 + \sqrt{C_2^2+4B_1B_3B_5}}, \\
\gamma_3 &= \frac{-C_3 + \sqrt{C_3^2 + 4B_2B_3B_6}}{2B_1} \quad \text{or} \quad \frac{2B_2B_6}{C_3 + \sqrt{C_3^2+4B_2B_3B_6}},
\end{aligned}$$

in the equation

$$C_4 = -B_5\frac{\gamma_1\gamma_3}{\gamma_2} + B_4B_6\frac{\gamma_2}{\gamma_1\gamma_3}$$

The result of this action is

$$\begin{aligned}
4B_1B_2B_3C_4 + C_1C_2C_3 &- C_1\sqrt{C_2^2+4B_1B_3B_5}\sqrt{C_3^2+4B_2B_3B_6} \\
&+ C_2\sqrt{C_3^2+4B_2B_3B_6}\sqrt{C_1^2+4B_1B_2B_4} \\
&- C_3\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_2^2+4B_1B_3B_5} = 0;
\end{aligned}$$

and similar equations can be got for C_3 in terms of C_1, C_2, C_4; for C_2 in terms of C_1, C_3, C_4; and for C_1 in terms of C_2, C_3, C_4.

7. On comparison of these results with those of Professor NANSON it will be found that instead of an explicit expression for $|a_1b_2c_3d_4|$ in terms of its coaxial minors, and an explicit expression for one of the coaxial minors of the 3rd order in terms of the three others and those of lower order, he obtains in each case an unsolved biquadratic equation. The presumption therefore is that each of his biquadratics must be resolvable into linear factors. This will now be shown to be the case. The series of necessary transformations is among the most interesting of the kind, and therefore well worthy of attention apart altogether from the problem with which they are here connected.

8. The latter of the two biquadratics is

$$\begin{vmatrix}
DL & CQ & BR & AQRL+2BCD \\
CP & DM & AR & BRPM+2CAD \\
BP & AQ & DN & CPQN+2ABD \\
AL & BM & CN & DLMN+2ABC
\end{vmatrix} = 0,$$

where

$$D, C, B, A;\ R, Q, L, P, M, N$$

correspond to but are not identical with the

$$C_1, C_2, C_3, C_4;\ B_1, B_2, B_3, B_4, B_5, B_6$$

of the present paper.

Now this determinant is easily seen to be the same as

$$\frac{1}{L^2M^2N^2}\begin{vmatrix} DLMN & CQMN & BRMN & AQRLMN+2BCDMN \\ CPNL & DMNL & ARNL & BRPMNL+2CADNL \\ BPLM & AQLM & DNLM & CPQNLM+2ABDLM \\ AL & BM & CN & DLMN\ +2ABC \end{vmatrix}.$$

Taking BC/L times each element of the last row from the corresponding element of the 1st row, CA/M times each element of the last row from the corresponding element of the 2nd row, and AB/N times each element of the last row from the corresponding element of the 3rd row, we transform this new determinant into

$$\begin{vmatrix} DLMN-ABC & CQMN-B^2C\frac{M}{L} & BRMN-BC^2\frac{N}{L} & AQRLMN+BCDMN-\frac{2AB^2C^2}{L} \\ CPNL-A^2C\frac{L}{M} & DLMN-ABC & ARNL-AC^2\frac{N}{M} & BRPLMN+CADLN-\frac{2A^2BC^2}{M} \\ BPLM-A^2B\frac{L}{M} & AQLM-AB^2\frac{M}{N} & DLMN-ABC & CPQLMN+ABDLM-\frac{2A^2B^2C}{N} \\ AL & BM & CN & DLMN+2ABC \end{vmatrix}.$$

Diminishing now each element of the last column by BC/L times the corresponding element of the 1st column, by CA/M times the corresponding element of the 2nd column, and by AB/N times the corresponding element of the 3rd column, we change the last column into

$$\left.\begin{matrix} AQRLMN-AC^2QN-AB^2RM+\frac{AB^2C^2}{L} \\ BRPLMN-BA^2RL-BC^2PN+\frac{A^2BC^2}{M} \\ CPQLMN-CB^2PM-CA^2QL+\frac{A^2B^2C}{N} \\ DLMN-ABC \end{matrix}\right\} \text{ or } \left\{\begin{matrix} \frac{A}{L}(NLQ-B^2)(LMR-C^2) \\ \frac{B}{M}(LMR-C^2)(MNP-A^2) \\ \frac{C}{N}(MNP-A^2)(NLQ-B^2) \\ DLMN-ABC \end{matrix}\right. ;$$

and if, merely for shortness' sake, we put

$$PL\left(1-\frac{A^2}{MNP}\right)=\lambda^2,$$
$$QM\left(1-\frac{B^2}{NLQ}\right)=\mu^2,$$
$$RN\left(1-\frac{C^2}{LMR}\right)=\nu^2,$$

the determinant becomes

$$\begin{vmatrix} DLMN-ABC & CN\mu^2 & BM\nu^2 & AL\mu^2\nu^2 \\ CN\lambda^2 & DLMN-ABC & AL\nu^2 & BM\nu^2\lambda^2 \\ BM\lambda^2 & AL\mu^2 & DLMN-ABC & CN\lambda^2\mu^2 \\ AL & BM & CN & DLMN-ABC \end{vmatrix}.$$

Dividing the columns by λ, μ, ν, $\lambda\mu\nu$ respectively, and multiplying the rows in order by the same, we obtain

$$\left|\begin{array}{cccc} \mathrm{DLMN}-\mathrm{ABC} & \mathrm{CN}\lambda\mu & \mathrm{BM}\nu\lambda & \mathrm{AL}\mu\nu \\ \mathrm{CN}\lambda\mu & \mathrm{DLMN}-\mathrm{ABC} & \mathrm{AL}\mu\nu & \mathrm{BM}\nu\lambda \\ \mathrm{BM}\nu\lambda & \mathrm{AL}\mu\nu & \mathrm{DLMN}-\mathrm{ABC} & \mathrm{CN}\lambda\mu \\ \mathrm{AL}\mu\nu & \mathrm{BM}\nu\lambda & \mathrm{CN}\lambda\mu & \mathrm{DLMN}-\mathrm{ABC} \end{array}\right| ,$$

—a determinant which is seen to have all the elements of the primary diagonal alike, all the elements of the secondary diagonal alike, and to be symmetric with respect to both diagonals. Such a determinant, when of the 4th order, must clearly be a function of the four elements which necessarily recur in every line; and, as a matter of fact, it is known to be expressible as the product of four factors, the first of which is the sum of the said four elements, and differs from each of the others in the sign of two of its last three terms. The biquadratic we began with is thus the same as

$$\begin{aligned} &(\mathrm{DLMN}-\mathrm{ABC}+\mathrm{CN}\lambda\mu+\mathrm{BM}\nu\lambda+\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}+\mathrm{CN}\lambda\mu-\mathrm{BM}\nu\lambda-\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}-\mathrm{CN}\lambda\mu+\mathrm{BM}\nu\lambda-\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}-\mathrm{CN}\lambda\mu-\mathrm{BM}\nu\lambda+\mathrm{AL}\mu\nu) \ = \ 0 ; \end{aligned}$$

so that if we put back the values of λ, μ, ν and solve, we have

$$\begin{aligned} \mathrm{D}=\frac{1}{\mathrm{LMN}}\{\mathrm{ABC} \pm \mathrm{C}\sqrt{\mathrm{A}^2-\mathrm{MNP}}\sqrt{\mathrm{B}^2-\mathrm{NLQ}} &\pm \mathrm{B}\sqrt{\mathrm{C}^2-\mathrm{LMR}}\sqrt{\mathrm{A}^2-\mathrm{MNP}} \\ &\pm \mathrm{C}\sqrt{\mathrm{A}^2-\mathrm{MNP}}\sqrt{\mathrm{B}^2-\mathrm{NLQ}}\}, \end{aligned}$$

and this, on the required changes being made, will be found to be identical with the result of section 6.

9. The other biquadratic referred to is

$$\left|\begin{array}{cccc} \theta & (1-\mathrm{C})\sqrt{1-\mathrm{B}^2} & (1-\mathrm{B})\sqrt{1-\mathrm{C}^2} & (1-\mathrm{A})\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} \\ (1-\mathrm{C})\sqrt{1-\mathrm{A}^2} & \theta & (1-\mathrm{A})\sqrt{1-\mathrm{C}^2} & (1-\mathrm{B})\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} \\ (1-\mathrm{B})\sqrt{1-\mathrm{A}^2} & (1-\mathrm{A})\sqrt{1-\mathrm{B}^2} & \theta & (1-\mathrm{C})\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} \\ 1-\mathrm{A} & 1-\mathrm{B} & 1-\mathrm{C} & \theta \end{array}\right| = 0 ,$$

where θ stands for $(\mathrm{A}-1)\ (\mathrm{B}-1)\ (\mathrm{C}-1)-\frac{1}{2}\Delta$. It is the biquadratic not for the general determinant $|a_1b_2c_3d_4|$ but for the very special instance

$$\left|\begin{array}{cccc} 1 & h & g^{-1} & 1 \\ h^{-1} & 1 & f & 1 \\ g & f^{-1} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right| \quad \text{or} \quad \Delta .$$

In this case the required transformation is very easy. All that is necessary is to divide the first three rows by $\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2}$, $\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2}$, $\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2}$ respectively, and then multiply in order the first three columns by the same. The result is

$$\begin{vmatrix} \theta & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} \\ (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & \theta & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} \\ (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & \theta & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} \\ (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & \theta \end{vmatrix} = 0,$$

where again the determinant has the elements of the primary diagonal all alike, the elements of the secondary diagonal all alike, and is symmetric with respect to both diagonals. As before, therefore, it resolves into four factors, and we have on substituting the value of θ

$$(A-1)(B-1)(C-1) - \tfrac{1}{2}\Delta \pm (1-C)\sqrt{1-A^2}\sqrt{1-B^2} \pm (1-B)\sqrt{1-C^2}\sqrt{1-A^2} \\ \pm (1-A)\sqrt{1-A^2}\sqrt{1-B^2} = 0,$$

or

$$\Delta = 2ABC - 2\Sigma AB + 2\Sigma A - 2 \pm 2(1-C)\sqrt{1-A^2}\sqrt{1-B^2} \pm 2(1-B)\sqrt{1-B^2}\sqrt{1-A^2} \\ \pm 2(1-A)\sqrt{1-A^2}\sqrt{1-B^2},$$

which is readily shown to be in agreement with the more general result in section 5.*

10. Not only does the determinant

$$\begin{vmatrix} DL & CQ & BR & AQRL+2BCD \\ CP & DM & AR & BRPM+2CAD \\ BP & AQ & DN & CPQN+2ABD \\ AL & BM & CN & DLMN+2ABC \end{vmatrix}$$

resolve into factors, but each of the two determinants into which it may be partitioned is also so resolvable. For, multiplying the columns in order by $\sqrt{MNQR}$, $\sqrt{NLRP}$, $\sqrt{LMPQ}$, 1, and then dividing the rows in order by $\sqrt{LQR}$, $\sqrt{MRP}$, $\sqrt{NPQ}$, $\sqrt{LMN}$, we obtain the new form

$$\begin{vmatrix} D\sqrt{LMN} & C\sqrt{NPQ} & B\sqrt{MRP} & A\sqrt{LQR}+\dfrac{2BCD}{\sqrt{LQR}} \\ C\sqrt{NPQ} & D\sqrt{LMN} & A\sqrt{LQR} & B\sqrt{MRP}+\dfrac{2CDA}{\sqrt{MRP}} \\ B\sqrt{MRP} & A\sqrt{LQR} & D\sqrt{LMN} & C\sqrt{NPQ}+\dfrac{2DAB}{\sqrt{NPQ}} \\ A\sqrt{LQR} & B\sqrt{MRP} & C\sqrt{NPQ} & D\sqrt{LMN}+\dfrac{2ABC}{\sqrt{LMN}} \end{vmatrix};$$

* Instead of the biquadratic of this section another might readily have been obtained from the single equation numbered (9) in Professor NANSON's paper, viz.,

$$\mu + \sqrt{1-B^2}\sqrt{1-C^2} + \sqrt{1-C^2}\sqrt{1-A^2} + \sqrt{1-A^2}\sqrt{1-B^2} = 0,$$

where

$$\mu = A+B+C+D-\tfrac{1}{2}\Delta-1-BC-CA-AB.$$

Observe also that this equation gives a much simpler expression for Δ, viz.:—

$$\Delta = -2+2\Sigma A-2\Sigma AB+2\Sigma\sqrt{1-B^2}\sqrt{1-C^2}.$$

and on partitioning this into two the first is seen to be

$$\begin{aligned}&= (D\sqrt{LMN}+C\sqrt{NPQ}+B\sqrt{MRP}+A\sqrt{LQR})\\ &\quad .(D\sqrt{LMN}+C\sqrt{NPQ}-B\sqrt{MRP}-A\sqrt{LQR})\\ &\quad .(D\sqrt{LMN}-C\sqrt{NPQ}+B\sqrt{MRP}-A\sqrt{LQR})\\ &\quad .(D\sqrt{LMN}-C\sqrt{NPQ}-B\sqrt{MRP}+A\sqrt{LQR}),\end{aligned}$$

and the second to be

$$\begin{vmatrix} D\sqrt{LMN} & C\sqrt{NPQ} & B\sqrt{MRP} & D\sqrt{LMN}\,.C\sqrt{NPQ}\,.B\sqrt{MRP} \\ C\sqrt{NPQ} & D\sqrt{LMN} & A\sqrt{LQR} & C\sqrt{NPQ}\,.D\sqrt{LMN}.A\sqrt{LQR} \\ B\sqrt{MRP} & A\sqrt{LQR} & D\sqrt{LMN} & B\sqrt{MRP}.A\sqrt{LQR}\,.D\sqrt{LMN} \\ A\sqrt{LQR} & B\sqrt{MRP} & C\sqrt{NPQ} & A\sqrt{LQR}\,.B\sqrt{MRP}\,.C\sqrt{NPQ} \end{vmatrix} \times \frac{2}{LMNPQR},$$

and therefore

$$\begin{aligned}&=(C\sqrt{NPQ}\,.A\sqrt{LQR}-D\sqrt{LMN}\,.B\sqrt{MRP})\\ &\quad(A\sqrt{LQR}\,.B\sqrt{MRP}-C\sqrt{NPQ}\,.D\sqrt{LMN})\\ &\quad(B\sqrt{MRP}.C\sqrt{NPQ}-A\sqrt{LQR}\,.D\sqrt{LMN}) \div \tfrac{1}{2}LMNPQR,\\ &=2(CAQ-DBM)(ABR-CDN)(BCP-ADL).\end{aligned}$$

11. Were it not for the divisor LMNPQR attached to the second determinant in the preceding section, the full determinant would be a function of only four variables, viz. :—

$$A\sqrt{LQR},\quad B\sqrt{MRP},\quad C\sqrt{NPQ},\quad D\sqrt{LMN};$$

and as a matter of fact the final expansion of it may be written

$$\begin{aligned}&\Sigma(A\sqrt{LQR})^4-2\Sigma(A\sqrt{LQR})^2(B\sqrt{MRP})^2\\ &\qquad+8(A\sqrt{LQR}\,.B\sqrt{MRP}\,.C\sqrt{NPQ}\,.D\sqrt{LMN})\\ &\qquad+\frac{4\Sigma(A\sqrt{LQR})^2(B\sqrt{MRP})^2(C\sqrt{NPQ})^2-4\Sigma(A\sqrt{LQR})^3.B\sqrt{MRP}.C\sqrt{NPQ}.D\sqrt{LMN}}{LMNPQR}.\end{aligned}$$

12. Standing in close connection with the subject-matter of the preceding sections—the connection of general with particular—is the problem of clearing the equation

$$x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab} = 0$$

of root-signs, or of transforming a fraction of which $x+h\sqrt{bc}+k\sqrt{ca}+l\sqrt{ab}$ is the denominator into one having its denominator rational. Viewing the matter in either way we reach the result

$$(x+h\sqrt{bc}+k\sqrt{ca}+l\sqrt{ab})(x+h\sqrt{bc}-k\sqrt{ca}-l\sqrt{ab})(x-h\sqrt{bc}+k\sqrt{ca}-l\sqrt{ab})(x-h\sqrt{bc}-k\sqrt{ca}+l\sqrt{ab})$$

or

$$\begin{aligned}&x^4+h^4b^2c^2+k^4c^2a^2+l^4a^2b^2\\ &\quad-2x^2(h^2bc+k^2ca+l^2ab)\\ &\quad-2abc(h^2k^2c+k^2l^2a+l^2h^2b)\\ &\quad-8xhklabc.\end{aligned}$$

This, however, is well known to be equal to the determinant

$$\begin{vmatrix} x & h\sqrt{bc} & k\sqrt{ca} & l\sqrt{ab} \\ h\sqrt{bc} & x & l\sqrt{ab} & k\sqrt{ca} \\ k\sqrt{ca} & l\sqrt{ab} & x & h\sqrt{bc} \\ l\sqrt{ab} & k\sqrt{ca} & h\sqrt{bc} & x \end{vmatrix};$$

and we may consequently say that *the rationalizant of the expression* $x+h\sqrt{bc}+k\sqrt{ca}+l\sqrt{ab}$ *is the biaxisymmetric determinant of the* 4*th order which has the terms of the expression for the elements of its first row, all the elements of its primary diagonal alike, and all the elements of its secondary diagonal alike.*

Another determinant form of the result is obtained by using the dialytic method of elimination. Taking the original equation and multiplying in succession by $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$, we have

$$\left.\begin{array}{l} x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab} = 0 \\ hbc + x\sqrt{bc} + lb\sqrt{ca} + kc\sqrt{ab} = 0 \\ kca + la\sqrt{bc} + x\sqrt{ca} + hc\sqrt{ab} = 0 \\ lab + ka\sqrt{bc} + hb\sqrt{ca} + x\sqrt{ab} = 0 \end{array}\right\},$$

and therefore on eliminating $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$ there results the rationalizant

$$\begin{vmatrix} x & h & k & l \\ hbc & x & lb & kc \\ kca & la & x & hc \\ lab & ka & hb & x \end{vmatrix}.$$

It is easy to change the one form into the other; indeed, this change is what has been effected in sections 8, 9, Professor NANSON having obtained his results in the latter of the two forms.

13. Another closely related problem, as Professor NANSON has made clear, is that of expressing $\cos(\alpha+\beta+\gamma)$ in terms of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, or say, for shortness' sake, S in terms of A, B, C.

Since

$$\cos(\alpha+\beta+\gamma) - \cos\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\sin\gamma + \cos\beta\sin\gamma\sin\alpha + \cos\gamma\sin\alpha\sin\beta = 0,$$

we have

$$\mathrm{S} - \mathrm{ABC} + \mathrm{A}\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} + \mathrm{B}\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} + \mathrm{C}\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} = 0,$$

and the problem is seen to be a case of the preceding, the result being either

$$\begin{vmatrix} \mathrm{S}-\mathrm{ABC} & \mathrm{A}\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} & \mathrm{B}\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} & \mathrm{C}\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} \\ \mathrm{A}\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} & \mathrm{S}-\mathrm{ABC} & \mathrm{C}\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} & \mathrm{B}\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} \\ \mathrm{B}\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} & \mathrm{C}\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} & \mathrm{S}-\mathrm{ABC} & \mathrm{A}\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} \\ \mathrm{C}\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} & \mathrm{B}\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} & \mathrm{A}\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} & \mathrm{S}-\mathrm{ABC} \end{vmatrix}$$

or

$$\begin{vmatrix} S-ABC & A & B & C \\ A(1-B^2)(1-C^2) & S-ABC & C(1-B^2) & B(1-C^2) \\ B(1-C^2)(1-A^2) & C(1-A^2) & S-ABC & A(1-C^2) \\ C(1-A^2)(1-B^2) & B(1-A^2) & A(1-B^2) & S-ABC \end{vmatrix},$$

which latter can be simplified, as Professor NANSON shows, into

$$\begin{vmatrix} S+2ABC & A & B & C \\ A+2BCS & S & C & B \\ B+2ACS & C & S & A \\ C+2ABS & B & A & S \end{vmatrix}.$$

This, however, can be obtained much more directly from the use of another expression for $\cos(\alpha+\beta+\gamma)$, viz.:—

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos(\beta+\gamma) + \cos\beta\cos(\gamma+\alpha) + \cos\gamma\cos(\alpha+\beta) - 2\cos\alpha\cos\beta\cos\gamma,$$

where nothing but cosines appears, the angles being

$$\alpha,\ \beta,\ \gamma;\quad \beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta;\quad \alpha+\beta+\gamma.$$

Making in this equation the substitutions

$$\begin{cases} \alpha = \alpha+\beta+\gamma, \\ \beta = -\gamma, \\ \gamma = -\beta, \end{cases} \begin{cases} \alpha = -\gamma, \\ \beta = \alpha+\beta+\gamma, \\ \gamma = -\alpha, \end{cases} \begin{cases} \alpha = -\beta, \\ \beta = -\alpha, \\ \gamma = \alpha+\beta+\gamma, \end{cases}$$

we obtain three other perfectly similar identities* connecting the same seven cosines, the complete set of four identities being in the notation above employed

$$\left.\begin{aligned} S + 2ABC - A\cos(\beta+\gamma) - B\cos(\gamma+\alpha) - C\cos(\alpha+\beta) &= 0 \\ A + 2SCB - S\cos(\beta+\gamma) - C\cos(\gamma+\alpha) - B\cos(\alpha+\beta) &= 0 \\ B + 2CSA - C\cos(\beta+\gamma) - S\cos(\gamma+\alpha) - A\cos(\alpha+\beta) &= 0 \\ C + 2BAS - B\cos(\beta+\gamma) - A\cos(\gamma+\alpha) - S\cos(\alpha+\beta) &= 0 \end{aligned}\right\}.$$

From these $\cos(\beta+\gamma)$, $\cos(\gamma+\alpha)$, $\cos(\alpha+\beta)$ can be eliminated, and the desired result at once obtained.

14. It may be noticed in passing that the substitution of $90^\circ-\alpha$, $90^\circ-\beta$, $90^\circ-\gamma$ for α, β, γ gives the similar relation between $\sin(\alpha+\beta+\gamma)$, $\sin\alpha$, $\sin\beta$, $\sin\gamma$.

It should also be noted that the corresponding expression for $\cos(\alpha+\beta)$ in terms of $\cos\alpha$ and $\cos\beta$ is obtained from an identity of a different type, viz., $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the set of equations being

$$\left.\begin{aligned} \sin\beta + \cos(\alpha+\beta).\sin\alpha - \cos\alpha\sin(\alpha+\beta) &= 0 \\ \cos(\alpha+\beta).\sin\beta + \sin\alpha - \cos\beta\sin(\alpha+\beta) &= 0 \\ \cos\alpha.\sin\beta + \cos\beta.\sin\alpha - \sin(\alpha+\beta) &= 0 \end{aligned}\right\}$$

* In effect the substitutions are the same as the circular substitution $\begin{pmatrix} S & A & B & C \\ A & B & C & S \end{pmatrix}$ if we consider $\cos(\beta+\gamma)$ $\cos(\gamma+\alpha)$, $\cos(\alpha+\beta)$ as invariant.

and the resulting equation *

$$\begin{vmatrix} 1 & \cos(\alpha+\beta) & \cos\alpha \\ \cos(\alpha+\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} = 0 .$$

The same identity almost suffices to give the corresponding relation between $\sin(\alpha+\beta)$, $\sin\alpha$, $\sin\beta$, the set of equations now being

$$\left.\begin{array}{rrrrl} & -\sin\beta\,.\cos\alpha & -\sin\alpha\,.\cos\beta & +\sin(\alpha+\beta) & = 0 \\ \sin\beta\,.\cos(\alpha+\beta) & & -\sin(\alpha+\beta)\,.\cos\beta & +\sin\alpha & = 0 \\ \sin\alpha\,.\cos(\alpha+\beta) & -\sin(\alpha+\beta)\,.\cos\alpha & & +\sin\beta & = 0 \\ \sin(\alpha+\beta)\,.\cos(\alpha+\beta) & -\sin\alpha\,.\cos\alpha & -\sin\beta\,.\cos\beta & +2\sin\alpha\sin\beta\sin(\alpha+\beta) & = 0 \end{array}\right\} ,$$

whence on the elimination of $\cos\alpha$, $\cos\beta$, $\cos(\alpha+\beta)$ we have

$$\begin{vmatrix} . & \sin\beta & \sin\alpha & \sin(\alpha+\beta) \\ \sin\beta & . & \sin(\alpha+\beta) & \sin\alpha \\ \sin\alpha & \sin(\alpha+\beta) & . & \sin\beta \\ \sin(\alpha+\beta) & \sin\alpha & \sin\beta & 2\sin\alpha\sin\beta\sin(\alpha+\beta) \end{vmatrix} = 0 .$$

15. The consideration of the relation between $\cos(\alpha+\beta+\gamma+\delta)$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$ leads at once to the question of the rationalization of the equation

$$a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw} = 0,$$

because

$$\cos(\alpha+\beta+\gamma+\delta) = \cos\alpha\cos\beta\cos\gamma\cos\delta - \Sigma\cos\gamma\cos\delta\sin\alpha\sin\beta + \sin\alpha\sin\beta\sin\gamma\sin\delta .$$

By proceeding in exactly the same manner as in section 12 the result of the rationalization is obtained in three forms, viz., (1) the product

$$\begin{aligned} &(a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw}) \\ .&(a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} - e\sqrt{yz} - f\sqrt{yw} - g\sqrt{zw} - h\sqrt{xyzw}) \\ .&(a + b\sqrt{xy} - c\sqrt{xz} - d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} - g\sqrt{zw} - h\sqrt{xyzw}) \\ .&(a + b\sqrt{xy} - c\sqrt{xz} - d\sqrt{xw} - e\sqrt{yz} - f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw}) \\ .&(a - b\sqrt{xy} + c\sqrt{xz} - d\sqrt{xw} + e\sqrt{yz} - f\sqrt{yw} + g\sqrt{zw} - h\sqrt{xyzw}) \\ .&(a - b\sqrt{xy} + c\sqrt{xz} - d\sqrt{xw} - e\sqrt{yz} + f\sqrt{yw} - g\sqrt{zw} + h\sqrt{xyzw}) \\ .&(a - b\sqrt{xy} - c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} - f\sqrt{yw} - g\sqrt{zw} + h\sqrt{xyzw}) \\ .&(a - b\sqrt{xy} - c\sqrt{xz} + d\sqrt{xw} - e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} - h\sqrt{xyzw}) , \end{aligned}$$

* It is interesting to note the mode in which the more general relation connecting $\cos(\alpha+\beta+\gamma)$, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, passes over into this on putting $\gamma=0$ in the former. The result of the substitution is

$$\begin{vmatrix} \cos\alpha & 1 & \cos(\alpha+\beta) & \cos\beta+2\cos\alpha\cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 & \cos\alpha+2\cos\beta\cos(\alpha+\beta) \\ 1 & \cos\alpha & \cos\beta & \cos(\alpha+\beta)+2\cos\alpha\cos\beta \\ \cos(\alpha+\beta) & \cos\beta & \cos\alpha & 1+2\cos\alpha\cos\beta\cos(\alpha+\beta) \end{vmatrix} ,$$

where the elements of the 4th column are easily transformed into zeros with the exception of the last element which becomes

$$1 + 2\cos\alpha\cos\beta\cos(\alpha+\beta) - \cos^2(\alpha+\beta) \quad \cos^2\beta - \cos^2\alpha,$$

so that the value of the determinant is seen to be

$$\begin{vmatrix} \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \\ 1 & \cos\alpha & \cos\beta \end{vmatrix}^2 .$$

With this mode of degeneration may be compared that seen on p. 377 of *Proc. Roy. Soc. Edin.*, xx

(2) the biaxisymmetric determinant

$$\begin{vmatrix}
a & b\sqrt{xy} & c\sqrt{xz} & d\sqrt{xw} & e\sqrt{yz} & f\sqrt{yw} & g\sqrt{zw} & h\sqrt{xyzw} \\
b\sqrt{xy} & a & e\sqrt{yz} & f\sqrt{yw} & c\sqrt{xz} & d\sqrt{xw} & h\sqrt{xyzw} & g\sqrt{zw} \\
c\sqrt{xz} & e\sqrt{yz} & a & g\sqrt{zw} & b\sqrt{xy} & h\sqrt{xyzw} & d\sqrt{xw} & f\sqrt{yw} \\
d\sqrt{xw} & f\sqrt{yw} & g\sqrt{zw} & a & h\sqrt{xyzw} & b\sqrt{xy} & c\sqrt{xz} & e\sqrt{yz} \\
e\sqrt{yz} & c\sqrt{xz} & b\sqrt{xy} & h\sqrt{xyzw} & a & g\sqrt{zw} & f\sqrt{yw} & d\sqrt{xw} \\
f\sqrt{yw} & d\sqrt{xw} & h\sqrt{xyzw} & b\sqrt{xy} & g\sqrt{zw} & a & e\sqrt{yz} & c\sqrt{xz} \\
g\sqrt{zw} & h\sqrt{xyzw} & d\sqrt{xw} & c\sqrt{xz} & f\sqrt{yw} & e\sqrt{yz} & a & b\sqrt{xy} \\
h\sqrt{xyzw} & g\sqrt{zw} & f\sqrt{yw} & e\sqrt{yz} & d\sqrt{xw} & c\sqrt{xz} & b\sqrt{xy} & a
\end{vmatrix},$$

and (3) the axisymmetric determinant

$$\begin{vmatrix}
a & b & c & d & e & f & g & h \\
bxy & a & ey & fy & cx & dx & hxy & g \\
cxz & ez & a & gz & bx & hxz & dx & f \\
dxw & fw & gw & a & hxw & bx & cx & e \\
eyz & cz & by & hyz & a & gz & fy & d \\
fyw & dw & hyw & by & gw & a & ey & c \\
gzw & hzw & dw & cz & fw & ez & a & b \\
hxyzw & gzw & fyw & eyz & dxw & cxz & bxy & a
\end{vmatrix}.$$

The third form is easily changed into the second by multiplying the columns in order by

$$1,\ \sqrt{xy},\ \sqrt{xz},\ \sqrt{xw},\ \sqrt{yz},\ \ldots\ldots,\ \sqrt{xyzw},$$

and then dividing the rows in order by the same. The mode of resolution of the second form into factors is well known.*

16. There is still another variant of the problem of sections 6, 8, viz., to express the relation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}w = 0$$

in purely algebraical form. In essence it is the same as the variant dealt with in section 13.

Subtracting $\cos^{-1}w$ from both sides, and then taking the cosines of the two equals, we have

$$xyz - x\sqrt{1-y^2}\sqrt{1-z^2} - y\sqrt{1-z^2}\sqrt{1-x^2} - z\sqrt{1-x^2}\sqrt{1-y^2} = w\,,$$

which is at once seen to be an equation of the form dealt with in section 12. The result of the rationalization is

$$\begin{vmatrix}
x & y & z & w+2xyz \\
y & x & w & z+2yxw \\
z & w & x & y+2zwx \\
w & z & y & x+2wzy
\end{vmatrix} = 0,$$

or

$$\Sigma x^4 - 2\Sigma x^2y^2 + 8xyzw + 4\Sigma x^2y^2z^2 - 4\,\Sigma\, x^3yzw = 0\,.$$

* See *Quart. Journ. of Math*, xviii. pp. 170, 171.

Similarly we have for the equation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 0$$

the purely algebraical equivalent

$$\begin{vmatrix} 1 & x & z \\ x & 1 & y \\ z & y & 1 \end{vmatrix} = 0;$$

and for the equation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}w + \cos^{-1}v = 0$$

a purely algebraical equivalent essentially the same as that referred to in section 15 as giving the relation between $\cos(\alpha+\beta+\gamma+\delta)$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$.

17. This suggests a very simple and perfectly symmetrical mode of expressing the relation of section 6 between the coaxial minors of an order lower than the fourth, viz.:—

$$\cos^{-1}\frac{C_1}{2\sqrt{-B_1B_2B_4}} + \cos^{-1}\frac{C_2}{2\sqrt{-B_1B_3B_5}} + \cos^{-1}\frac{C_3}{2\sqrt{-B_2B_3B_6}} + \frac{C_4}{2\sqrt{-B_4B_5B_6}} = 0.$$

The law of formation of the denominators is perhaps not clear, but this is due merely to a defect in the notation. If we substitute for B's and C's their values as given in terms of the coaxial minors of $|a_1b_2c_3d_4|$ we have

$$\sum \cos^{-1}\frac{|a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| - c_3|a_1b_2| + 2a_1b_2c_3}{2(-1)^{\frac{1}{2}}(|a_1b_2| - a_1b_2)^{\frac{1}{2}}(|a_1c_3| - a_1c_3)^{\frac{1}{2}}(|b_2c_3| - b_2c_3)^{\frac{1}{2}}} = 0;$$

and, further, if we denote by

$$\left| \begin{matrix} a_1b_2c_3 \\ 000 \end{matrix} \right|$$

the determinant got from $|a_1b_2c_3|$ by changing the elements of the primary diagonal into zeros, the relation may be written

$$\sum \cos^{-1}\frac{\left| \begin{matrix} a_1b_2c_3 \\ 000 \end{matrix} \right|}{2\left\{-\left| \begin{matrix} a_1b_2 \\ 00 \end{matrix} \right| \left| \begin{matrix} a_1c_3 \\ 00 \end{matrix} \right| \left| \begin{matrix} b_2c_3 \\ 00 \end{matrix} \right|\right\}^{\frac{1}{2}}} = 0.$$

18. Another matter which has light thrown upon it by certain of the preceding paragraphs is SYLVESTER's original illustration of the dialytic method of elimination as applied to ternary quadrics. It will be remembered that from the equations

$$\left.\begin{aligned} Bx^2 - 2C'xy + Ay^2 &= 0 \\ Cy^2 - 2A'yz + Bz^2 &= 0 \\ Az^2 - 2B'zx + Cx^2 &= 0 \end{aligned}\right\}$$

he deduced three others

$$\left.\begin{aligned} C'z^2 + Cxy - A'zx - B'yz &= 0 \\ A'x^2 + Ayz - B'xy - C'zx &= 0 \\ B'y^2 + Bzx - C'yz - A'xy &= 0 \end{aligned}\right\},$$

and thus obtained the eliminant in the form

$$\begin{vmatrix} . & C & B & -2A' & . & . \\ C & . & A & . & -2B' & . \\ B & A & . & . & . & -2C' \\ A' & . & . & A & -C' & -B' \\ . & B' & . & -C' & B & -A' \\ . & . & C' & -B' & -A' & C \end{vmatrix}$$

which, it was afterwards shown,

$$= 2 \begin{vmatrix} A & C' & B' \\ C' & B & A' \\ B' & A' & C \end{vmatrix}^2 .$$

Now the given equations may be written

$$\left.\begin{aligned} \frac{y\sqrt{A}}{x\sqrt{B}} + \frac{x\sqrt{B}}{y\sqrt{A}} &= \frac{2C'}{\sqrt{AB}} \\ \frac{z\sqrt{B}}{y\sqrt{C}} + \frac{y\sqrt{C}}{z\sqrt{B}} &= \frac{2A'}{\sqrt{BC}} \\ \frac{x\sqrt{C}}{z\sqrt{A}} + \frac{z\sqrt{A}}{x\sqrt{C}} &= \frac{2B'}{\sqrt{CA}} \end{aligned}\right\} ;$$

consequently it is seen that there exists the relation

$$\cos^{-1}\frac{A'}{\sqrt{BC}} + \cos^{-1}\frac{B'}{\sqrt{CA}} + \cos^{-1}\frac{C'}{\sqrt{AB}} = 0 ,$$

and therefore

$$\begin{vmatrix} 1 & \dfrac{A'}{\sqrt{BC}} & \dfrac{C'}{\sqrt{AB}} \\ \dfrac{A'}{\sqrt{BC}} & 1 & \dfrac{B'}{\sqrt{CA}} \\ \dfrac{C'}{\sqrt{AB}} & \dfrac{B'}{\sqrt{CA}} & 1 \end{vmatrix} = 0 .$$

Similarly the resultant of

$$\left.\begin{aligned} Bx^2 - Dxy + Ay^2 &= 0 \\ Cy^2 - Eyz + Bz^2 &= 0 \\ Lz^2 - Kzw + Cw^2 &= 0 \\ Aw^2 - Gwx + Lx^2 &= 0 \end{aligned}\right\}$$

is

$$\cos^{-1}\frac{D}{2\sqrt{AB}} + \cos^{-1}\frac{E}{2\sqrt{BC}} + \cos^{-1}\frac{K}{2\sqrt{CA}} + \cos^{-1}\frac{G}{2\sqrt{LA}} = 0 ,$$

and therefore from section 16 is

$$\begin{vmatrix} \dfrac{D}{2\sqrt{AB}} & \dfrac{E}{2\sqrt{BC}} & \dfrac{K}{2\sqrt{CL}} & \dfrac{G}{2\sqrt{LA}} + \dfrac{DEK}{4BC\sqrt{LA}} \\ \dfrac{E}{2\sqrt{BC}} & \dfrac{D}{2\sqrt{AB}} & \dfrac{G}{2\sqrt{LA}} & \dfrac{K}{2\sqrt{CL}} + \dfrac{EDG}{4AB\sqrt{CL}} \\ \dfrac{K}{2\sqrt{CL}} & \dfrac{G}{2\sqrt{LA}} & \dfrac{D}{2\sqrt{AB}} & \dfrac{E}{2\sqrt{BC}} + \dfrac{KGD}{4LA\sqrt{BC}} \\ \dfrac{G}{2\sqrt{LA}} & \dfrac{K}{2\sqrt{CL}} & \dfrac{E}{2\sqrt{BC}} & \dfrac{D}{2\sqrt{AB}} + \dfrac{GKE}{4CL\sqrt{AB}} \end{vmatrix} = 0 .$$

Multiplying the rows in order by $4BC\sqrt{LA}$, $4AB\sqrt{CL}$, $4LA\sqrt{BC}$, $4CL\sqrt{AB}$ we change this determinant into

$$\begin{vmatrix} 2DC\sqrt{BL} & 2E\sqrt{ABCL} & 2KB\sqrt{AC} & 2GBC+DEK \\ 2EA\sqrt{BL} & 2D\sqrt{ABCL} & 2GB\sqrt{AC} & 2KAB+EDG \\ 2KA\sqrt{BL} & 2G\sqrt{ABCL} & 2DL\sqrt{AC} & 2ELA+KGD \\ 2GC\sqrt{BL} & 2K\sqrt{ABCL} & 2EL\sqrt{AC} & 2DCL+GKE \end{vmatrix},$$

and now dividing the columns in order by $2\sqrt{BL}$, $2\sqrt{ABCL}$, $2\sqrt{AC}$, 1 we have finally

$$\begin{vmatrix} DC & E & KB & DEK+2GBC \\ EA & D & GB & EDG+2KAB \\ KA & G & DL & KGD+2ELA \\ GC & K & EL & GKE+2DCL \end{vmatrix} = 0,$$

which agrees with what has been obtained otherwise.*

* *Proc. Roy. Soc. Edin.*, xxi. p. 333.